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On functions of bounded semivariation

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Abstract

The concept of bounded variation has been generalized in many ways. In the frame of functions taking values in Banach space, the concept of bounded semivariation is a very important generalization. The aim of this paper is to provide an accessible summary on this notion, to illustrate it with an appropriate body of examples, and to outline its connection with the integration theory due to Kurzweil.

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1 Introduction

Different notions of variation appear when dealing with problems in infinite dimension. Among them, the semivariation is very frequent, being commonly found in studies involving convolution, Stieltjes type integration, and also in topics related to vector measures.

Initially called w-property, the concept of bounded semivariation for operator-valued functions was introduced in 1936 by M. Gowurin in his paper on the Stieltjes integral in Banach space [17]. Some decades later, the Gowurin w-property revealed to be very useful in the investigation of integral representations of continuous linear transformations (see [41] and [13]).

Nowadays, a handful of papers make use of the concept of bounded semivariation. However, in its majority, the results on such type of variation are only stated with no proofs or no proper references. Besides that, we can observe in the literature a lack of material collecting basic results on such a concept.

In view of this, the purpose of this survey is to summarize the present knowledge on semivariation. The presentation does not reflect the chronological order of the discoveries, but rather attempts to organize results in a logical framework. Moreover, in order to make

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these notes self-contained, most results are presented with a detailed proof and some illustrative examples are given. We trust that our citations and bibliography sufficiently identify the appropriate antecedent.

This survey includes, besides basic results and properties, also a section dedicated to the investigation of the relation between semivariation and non-absolute integrals.

First, let us fix some notation.

Throughout this survey X and Y denote Banach spaces and $L(X,Y)$ stands for the Banach space of bounded linear operators from X to Y. By $\|\cdot\|_X$ and $\|\cdot\|_{L(X,Y)}$ we denote the norm in X and the usual operator norm in $L(X,Y)$, respectively. In particular, we write $L(X) = L(X,X)$ and $X^* = L(X,\mathbb{R})$.

For an arbitrary function $f : [a,b\,] \to X$ we set $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|_X$.

Consider a nondegenerate closed interval $[a,b]$ and denote by $\mathcal{D}[a,b]$ the set of all finite divisions of $[a,b]$ of the form

$$D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}, \quad a = \alpha_0 < \alpha_1 < \ldots < \alpha_{\nu(D)} = b\,,$$

where $\nu(D) \in \mathbb{N}$ corresponds to the number of subintervals in which $[a,b]$ is divided.

With these concepts in hand we are ready to define the semivariation of an operator-valued function.

Definition 1.1. Given a function $F : [a,b\,] \to L(X,Y)$ and a division $D \in \mathcal{D}[a,b\,]$, let

$$V(F,D,[a,b\,]) = \sup\left\{ \left\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, x_j \right\|_Y \,:\, x_j \in X,\ \|x_j\|_X \le 1 \right\}.$$

The *semivariation* of F on $[a,b\,]$ is then defined by

$$\mathrm{SV}_a^b(F) = \sup\{V(F,D,[a,b\,]) \,:\, D \in \mathcal{D}[a,b\,]\}.$$

If $\mathrm{SV}_a^b(F) < \infty$, we say that the function F is of bounded semivariation on $[a,b\,]$. The set of all functions $F : [a,b\,] \to L(X,Y)$ of bounded semivariation on $[a,b\,]$ we denote by $SV([a,b\,], L(X,Y))$.

If no misunderstanding can arise, we write simply $V(F,D)$ instead of $V(F,D,[a,b\,])$.

Remark 1.2. The concept presented in Definition 1.1, called w-property in [17], is also known as $(\mathcal{B})$-variation, with respect to the bilinear triple $\mathcal{B} = (L(X,Y),X,Y)$. For details, see [33] and [14]. The terminology used in this paper is consistent with that found in the book by Hönig [22] and seem to be the most frequent in literature. However, we call the readers attention to the fact that the term 'semivariation' might also appear with slight different formulation - for example, when applied to measure theory or to functions with values in a general Banach space. See, for instance, [7], [9] or, in the frames of functions with values in locally convex spaces, [11].

It is not hard to see that the semivariation is more general than the notion of variation in the sense of Jordan. Indeed, for $F : [a, b\,] \to L(X, Y)$, we have

$$\mathrm{SV}_a^b(F) \le \mathrm{var}_a^b(F)$$

where $\mathrm{var}_a^b(F)$ stands for the variation of F on $[a, b\,]$ and is given by

$$\mathrm{var}_a^b(F) = \sup\left\{ \sum_{j=1}^{\nu(D)} \|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} \,:\, D \in \mathcal{D}[a, b\,] \right\}.$$

Denoting by $BV([a, b\,], L(X, Y))$ the set of all functions $F : [a, b\,] \to L(X, Y)$ of bounded variation on $[a, b\,]$ (i.e $\mathrm{var}_a^b(F) < \infty$), clearly,

$$BV([a, b\,], L(X, Y)) \subseteq SV([a, b\,], L(X, Y)).$$

The relation between these two sets will be analysed in more details in Section 4.

The following example of a function of bounded semivariation was inspired by some ideas found in [40].

Example 1.3. Let ℓ_2 be the Banach space of sequences $x = \{x_n\}_n$ in $\mathbb{R}$ such that the series $\sum_{n=1}^{\infty} |x_n|^2$ converges, equipped with the norm

$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$$

Denote by e_k, $k \in \mathbb{N}$, the canonical Schauder basis of ℓ_2, where e_k is the sequence whose k-th term is 1 and all other terms are zero.

For each $k \in \mathbb{N}$, consider $y_k \in \ell_2$ given by $y_k = \frac{1}{k} e_k$, that is,

$$y_k = \{y_n^{(k)}\}_n \;\text{ with }\; y_k^{(k)} = \frac{1}{k} \;\text{ and }\; y_n^{(k)} = 0 \;\text{ for }\; n \neq k.$$

Note that the series $\sum_{k=1}^{\infty} y_k$ converges in ℓ_2 and denote by S its sum.

Let $F : [0, 1] \to L(\mathbb{R}, \ell_2)$ be given by

$$\big(F(t)\big)\, x = \begin{cases} x \displaystyle\sum_{k=1}^{n} y_k & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}],\; n \in \mathbb{N}, \\ x\, S & \text{if } t = 0 \end{cases}$$

for $t \in [0, 1]$ and $x \in \mathbb{R}$.

In order to prove that $F \in SV([0, 1], L(\mathbb{R}, \ell_2))$, let us consider $D \in \mathcal{D}[0, 1]$ with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$. Put

$$n_j = \max\{k \in \mathbb{N} \,:\, k\alpha_j \le 1\} \;\text{ for }\; j = 1, \ldots, \nu(D),$$

and $\Lambda = \{j : n_j < n_{j-1}\} \subset \{2, \dots, \nu(D)\}$. For $x_j \in \mathbb{R}$, $j = 1, \dots, \nu(D)$ with $|x_j| \le 1$ we have $F(\alpha_j)x_j = \sum_{k=1}^{n_j} y_k$ and consequently

$$\begin{aligned}\sum_{j=1}^{\nu(D)}[F(\alpha_j) - F(\alpha_{j-1})]x_j &= x_1\Big(\sum_{k=1}^{n_1} y_k - S\Big) + \sum_{j=2}^{\nu(D)} x_j\Big(\sum_{k=1}^{n_j} y_k - \sum_{m=1}^{n_{j-1}} y_m\Big)\\ &= -x_1\Big(\sum_{k=n_1+1}^{\infty} y_k\Big) - \sum_{j\in\Lambda} x_j\Big(\sum_{k=n_j+1}^{n_{j-1}} y_k\Big)\end{aligned}$$

For $k \in \mathbb{N}$, define

$$\lambda_k = \begin{cases} -x_j & \text{if } \; n_j < k \le n_{j-1}, \quad j \in \Lambda \\ -x_1 & \text{if } \; k > n_1 \\ 0 & \text{otherwise} \end{cases} . \tag{1.1}$$

Using this sequence and the definition of y_k, we can write

$$\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j) - F(\alpha_{j-1})]\,x_j\Big\|_2^2 = \Big\|\sum_{k=1}^{\infty} \lambda_k\, y_k\Big\|_2^2 = \sum_{k=2}^{\infty}\Big|\frac{\lambda_k}{k}\Big|^2 \le \sum_{k=2}^{\infty}\frac{1}{k^2}.$$

This shows that $\mathrm{SV}_0^1(F) \le \Big(\sum_{k=2}^{\infty}\frac{1}{k^2}\Big)^{1/2} = \sqrt{\frac{\pi^2}{6} - 1}$

We claim that $\mathrm{SV}_0^1(F) = \sqrt{\frac{\pi^2}{6} - 1}$. Indeed, fixed an arbitrary $N \in \mathbb{N}$, consider $D_N \in \mathcal{D}[0,1]$ given by

$$D_N = \Big\{0, \frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}, 1\Big\},$$

Thus, for $x_j \in \mathbb{R}$, $j = 1, \dots, N$ with $|x_j| \le 1$ we have

$$\begin{aligned}&\Big\|\sum_{\ell=1}^{N-1}[F(\tfrac{1}{\ell}) - F(\tfrac{1}{\ell+1})]x_\ell + [F(\tfrac{1}{N}) - F(0)]x_N\Big\|_2\\ &\qquad = \Big\|\sum_{\ell=1}^{N-1} x_\ell y_{\ell+1} + \sum_{k=N+1}^{\infty} x_N y_k\Big\|_2 = \Big(\sum_{k=2}^{\infty}\Big|\frac{\tilde{x}_k}{k}\Big|^2\Big)^{1/2}\end{aligned}$$

where $\tilde{x}_k = x_{k-1}$ if $k = 2, \dots, N$, and $\tilde{x}_k = x_N$ for $k \in \mathbb{N}$, $k > N$. Taking the supremum over all possible choices of $x_j \in \mathbb{R}$, $j = 1, \dots, N$ with $|x_j| \le 1$ we obtain

$$V(F, D_N, [0,1]) = \Big(\sum_{k=2}^{\infty}\frac{1}{k^2}\Big)^{1/2},$$

which proves the claim.

Remark 1.4. In the particular case $X = \mathbb{R}$ the space $SV([a,b], L(\mathbb{R}, Y))$ can be regarded as the space of the functions of weak bounded variation, usually denoted by $BW([a,b], Y)$ (c.f. [20]). This is clear once we recall that the weak variation of a function $f : [a,b] \to Y$ is given by $\mathrm{W}_a^b(f) = \sup\{W(f,D) : D \in \mathcal{D}[a,b]\}$ where

$$W(f,D) = \sup\left\{ \Big\| \sum_{j=1}^{\nu(D)} \big[f(\alpha_j) - f(\alpha_{j-1})\big]\, \lambda_j \Big\|_Y \ : \ \lambda_j \in \mathbb{R},\ |\lambda_j| \le 1 \right\} \ \text{for} \ \ D \in \mathcal{D}[a,b].$$

Therefore, we can say that the semivariation of the function $F : [0,1] \to L(\mathbb{R}, \ell_2)$ in the Example 1.3 coincides with the weak variation of $f : [0,1] \to \ell_2$ given by $f(t) = (F(t))1$ for $t \in [0,1]$.

2 Semivariation: basic results

This section summarizes basic properties of the semivariation that are often mentioned without proof in papers which are directly or indirectly connected to such a notion. In order to make this work as complete as possible, all the proofs are included. Most of the results can be found, for instance, in [20], [22] and [37].

We start by noting that $SV([a,b], L(X,Y))$ is a vector space.

Proposition 2.1. *Let $F, G \in SV([a,b], L(X,Y))$ and $\lambda \in \mathbb{R}$ be given. Then both functions $(F+G)$ and (λF) are of bounded semivariation on $[a,b]$, and*

$$\mathrm{SV}_a^b(F+G) \le \mathrm{SV}_a^b(F) + \mathrm{SV}_a^b(G) \quad \textit{and} \quad \mathrm{SV}_a^b(\lambda F) = |\lambda|\ \mathrm{SV}_a^b(F). \tag{2.1}$$

Proof. The assertions follow from the fact that the relations

$$V(F+G,D) \le V(F,D) + V(G,D) \quad \text{and} \quad V(\lambda F, D) = |\lambda|\, V(F,D)$$

hold for every division $D \in \mathcal{D}[a,b]$. □

According to (2.1), $\mathrm{SV}_a^b(\,\cdot\,)$ defines a seminorm on the space of functions of bounded semivariation. On the other hand, if we put

$$\|F\|_{SV} = \|F(a)\|_{L(X,Y)} + \mathrm{SV}_a^b(F) \quad \text{for} \ \ F \in SV([a,b], L(X,Y), \tag{2.2}$$

then $SV([a,b], L(X,Y))$ becomes a normed space. This fact is a consequence of (2.1) together with the following assertion.

Proposition 2.2. *Let $F \in SV([a,b], L(X,Y))$. Then $\mathrm{SV}_a^b(F) = 0$ if and only if $F \equiv C$ for some fixed operator $C \in L(X,Y)$.*

Proof. Clearly, the semivariation of a constant function is zero. Conversely, assume that $\mathrm{SV}_a^b(F) = 0$. Given $t \in (a, b]$, if we consider the division $D = \{a, t, b\}$ of $[a, b\,]$, for any $x \in X$ with $\|x\|_X \leq 1$ we have

$$\|F(t)x - F(a)x\|_Y = \big\|[F(t) - F(a)]x + [F(b) - F(t)]0\big\|_Y \leq V(F, D, [a, b\,]).$$

Therefore $[F(t) - F(a)] = 0 \in L(X, Y)$, that is, F is a constant function. ☐

Remark 2.3. It is worth mentioning that in the definition of the norm $\|\cdot\|_{SV}$ we can use the fixed value of the function in any point of the interval, that is, taking $c \in [a, b\,]$, we can consider

$$\|F\|_{SV} = \|F(c)\|_{L(X,Y)} + \mathrm{SV}_a^b(F), \quad F \in SV([a, b\,], L(X, Y)).$$

The choice of the left-ending point of the interval seems to be the most common in the literature, though. Therefore, in this work, we assume the norm in $SV([a, b\,], L(X, Y))$ as introduced in (2.2).

Note that, for $F : [a, b\,] \to L(X, Y)$ and $t \in [a, b\,]$ we have

$$\|F(t)\|_{L(X,Y)} \leq \|F(a)\|_{L(X,Y)} + \mathrm{SV}_a^b(F).$$

Hence, every function $F \in SV([a, b\,], L(X, Y))$ is bounded and

$$\|F\|_\infty \leq \|F\|_{SV}.$$

In view of this, we can say that the topology induced in the space $SV([a, b\,], L(X, Y))$ by the supremum norm is weaker than the one induced by $\|\cdot\|_{SV}$.

In the sequel we prove that the space of functions of bounded semivariation is complete when equipped with the norm $\|\cdot\|_{SV}$ (c.f. [37, Proposition 4] or [20, I.3.3]). To this aim, we will need the following convergence result.

Lemma 2.4. *Let $F : [a, b\,] \to L(X, Y)$, a sequence $\{F_n\}_n \subset SV([a, b\,], L(X, Y))$ and a constant $M > 0$ be such that*

$$\mathrm{SV}_a^b(F_n) \leq M \quad \textit{for every} \;\; n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \|F_n(t)x - F(t)x\|_Y = 0 \quad \textit{for every} \;\; t \in [a, b\,] \;\; \textit{and} \;\; x \in X.$$

Then $\mathrm{SV}_a^b(F) \leq M$.

Proof. Let $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}$ be a division of $[a,b\,]$ and let $x_j\in X$, $j=1,\dots,\nu(D)$ with $\|x_j\|_X\le 1$. Note that, for each $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\,x_j\Big\|_Y\\
&\quad\le\Big\|\sum_{j=1}^{\nu(D)}[F_n(\alpha_j)-F_n(\alpha_{j-1})]\,x_j\Big\|_Y+\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F_n(\alpha_j)-F(\alpha_{j-1})+F_n(\alpha_{j-1})]\,x_j\Big\|_Y\\
&\quad\le M+\sum_{j=1}^{\nu(D)}\|[F(\alpha_j)-F_n(\alpha_j)]x_j\|_Y+\sum_{j=1}^{\nu(D)}\|[F(\alpha_{j-1})-F_n(\alpha_{j-1})]x_j\|_Y
\end{aligned}\tag{2.3}$$

Given $\varepsilon>0$, there is $N_D\in\mathbb{N}$ such that

$$\|[F(\alpha_j)-F_{N_D}(\alpha_j)]x_i\|_Y<\frac{\varepsilon}{2\,\nu(D)}\quad\text{for}\ \ j=0,1,\dots,\nu(D),$$

where $i=j,j+1$ (whenever it has a sense). Therefore, taking $n=N_D$ in (2.3) we obtain

$$\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\,x_j\Big\|_Y<M+\varepsilon$$

which implies that

$$V(F,D)\le M+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it follows that $V(F,D)\le M$ for every $D\in\mathcal{D}[a,b\,]$, and consequently $\mathrm{SV}_a^b(F)\le M$. □

The previous convergence result usually appears applied to some integration theory This type of result, often mentioned as Helly-Bray theorem (cf. [22, Theorem I.5.8] or [10]), will be study in Section 5 in the frames of Kurzweil-Stieltjes integral.

Now, we are ready to prove the completeness of the space $SV([a,b\,],L(X,Y))$.

Theorem 2.5. *$SV([a,b\,],L(X,Y))$ is a Banach space with respect to the norm $\|\cdot\|_{SV}$.*

Proof. Let $\{F_n\}_n$ be a Cauchy sequence in $SV([a,b\,],L(X,Y))$ with respect to the norm $\|\cdot\|_{SV}$. This means that given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\|F_n(t)-F_m(t)\|_{L(X,Y)}\le\|F_n-F_m\|_{SV}<\varepsilon,\quad n,\,m\ge n_0\ \ \text{and}\ \ t\in[a,b\,].\tag{2.4}$$

Hence, for each $t\in[a,b\,]$, $\{F_n(t)\}_n$ is a Cauchy sequence in $L(X,Y)$ which implies that there exists $F(t)\in L(X,Y)$ such that

$$\lim_{n\to\infty}\|F_n(t)-F(t)\|_{L(X,Y)}=0.$$

Moreover, due to (2.4), this convergence is uniform on $[a,b]$. By the fact that $\{F_n\}_n$ is a Cauchy sequence there exists $M>0$ such that $\mathrm{SV}_a^b(F_n)\le M$ for every $n\in\mathbb{N}$. Therefore, by Lemma 2.4 $F\in SV([a,b],L(X,Y))$.

It remains to show that the convergence is true also in the topology induced by the norm $\|\cdot\|_{SV}$. To this aim, consider a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}$ of $[a,b]$ and arbitrary $x_j\in X$, $j=1,\dots,\nu(D)$ with $\|x_j\|_X\le 1$. By (2.4), for n, $m\ge n_0$, we have

$$\Big\|\sum_{j=1}^{\nu(D)}[F_n(\alpha_j)-F_m(\alpha_j)-F_n(\alpha_{j-1})+F_m(\alpha_{j-1})]\,x_j\Big\|_Y<\varepsilon.$$

Thus taking the limit $m\to\infty$ we obtain

$$\Big\|\sum_{j=1}^{\nu(D)}[F_n(\alpha_j)-F(\alpha_j)-F_n(\alpha_{j-1})+F(\alpha_{j-1})]\,x_j\Big\|_Y\le\varepsilon,$$

that is, $V((F_n-F),D)\le\varepsilon$, for $n\ge n_0$. Since the division $D\in\mathcal{D}[a,b]$ is arbitrary, it follows that $\lim_{n\to\infty}\mathrm{SV}(F_n-F)=0$, concluding the proof. □

The following theorem proves that the functions of bounded variation are multipliers for the space $SV([a,b],L(X,Y))$ (see [22, Lemma I.1.11]).

Theorem 2.6. *Let $F\in SV([a,b],L(X,Y))$ and $G\in BV([a,b],L(X))$. Consider the function $FG:[a,b]\to L(X,Y)$ given by $(FG)(t)=F(t)\,G(t)$ for $t\in[a,b]$. Then $F\,G\in SV([a,b],L(X,Y))$ and*

$$\mathrm{SV}_a^b(F\,G)\le\|F\|_\infty\,\mathrm{var}_a^b(G)+\|G\|_\infty\,\mathrm{SV}_a^b(F).$$

Proof. Consider a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}$ of $[a,b]$ and let $x_j\in X$, $j=1\dots,\nu(D)$ with $\|x_j\|_X\le 1$. Therefore

$$\begin{aligned}
&\Big\|\sum_{j=1}^{\nu(D)}[(F\,G)(\alpha_j)-(F\,G)(\alpha_{j-1})]\,x_j\Big\|_Y\\
&\quad=\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)\,G(\alpha_j)-F(\alpha_j)\,G(\alpha_{j-1})+F(\alpha_j)\,G(\alpha_{j-1})-F(\alpha_{j-1})\,G(\alpha_{j-1})]\,x_j\Big\|_Y\\
&\quad\le\Big\|\sum_{j=1}^{\nu(D)}F(\alpha_j)[G(\alpha_j)-G(\alpha_{j-1})]\,x_j\Big\|_Y+\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\,G(\alpha_{j-1})\,x_j\Big\|_Y\\
&\quad\le\|F\|_\infty\sum_{j=1}^{\nu(D)}\|G(\alpha_j)-G(\alpha_{j-1})\|_{L(X)}+\|G\|_\infty\left\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\frac{G(\alpha_{j-1})\,x_j}{\|G\|_\infty}\right\|_Y\\
&\quad\le\|F\|_\infty\,\mathrm{var}_a^b(G)+\|G\|_\infty\,\mathrm{SV}_a^b(F).
\end{aligned}$$

This implies that

$$V((FG), D) \leq \|F\|_\infty \operatorname{var}_a^b(G) + \|G\|_\infty \operatorname{SV}_a^b(F)$$

for every $D \in \mathcal{D}[a, b\,]$, wherefrom the result follows. $\square$

The next theorem presents some algebraic properties of the semivariation.

Theorem 2.7. *If $F : [a, b\,] \to L(X, Y)$ and $[c, d] \subset [a, b\,]$, then*

$$\operatorname{SV}_c^d(F) \leq \operatorname{SV}_a^b(F).$$

Moreover,

$$\operatorname{SV}_a^b(F) \leq \operatorname{SV}_a^c(F) + \operatorname{SV}_c^b(F) \quad \textit{for} \;\; c \in [a, b\,]. \tag{2.5}$$

Proof. It is easy to see that, for every division D of $[c, d]$, taking $\tilde{D} = D \cup \{a, b\}$, we have $\tilde{D} \in \mathcal{D}[a, b\,]$ and

$$V(F, D, [c, d]) \leq V(F, \tilde{D}, [a, b\,]) \leq \operatorname{SV}_a^b(F),$$

Therefore $\operatorname{SV}_c^d(F) \leq \operatorname{SV}_a^b(F)$.

To prove the superadditivity, given $c \in [a, b\,]$ and an arbitray division $D \in \mathcal{D}[a, b\,]$, consider $D_1 = (D \cap [a, c]) \cup \{c\}$ and $D_2 = (D \cap [c, b]) \cup \{c\}$. Clearly, D_1 and D_2 are divisions of $[a, c]$ and $[c, b]$, respectively. In addition,

$$V(F, D, [a, b\,]) \leq V_a^b(F, D \cup \{c\}, [a, b\,]) \leq V(F, D_1, [a, c]) + V(F, D_2, [c, b]).$$

Hence

$$V(F, D, [a, b\,]) \leq \operatorname{SV}_a^c(F) + \operatorname{SV}_c^b(F) \quad \text{for every} \;\; D \in \mathcal{D}[a, b\,],$$

which leads to the inequality (2.5). $\square$

According to the previous theorem: *if $F \in SV([a, b\,], L(X, Y))$, then F is of bounded semivariation on each closed subinterval of $[a, b\,]$.* As a consequence we have the following.

Corollary 2.8. *Let $F \in SV([a, b\,], L(X, Y))$ be given. Then*

1. *the mapping $t \in [a, b\,] \longmapsto \operatorname{SV}_a^t(F)$ is nondecreasing;*
2. *the mapping $t \in [a, b\,] \longmapsto \operatorname{SV}_t^b(F)$ is nonincreasing.*

Theorem 2.7 indicates that, unlike the variation, the semivariation need not be additive with respect to intervals. Next example shows that the inequality in (2.5) may be strict.

Example 2.9. Let $F : [0, 1] \to L(\mathbb{R}, \ell_2)$ be the function given on Example 1.3, that is, for $t \in [0, 1]$ and $x \in \mathbb{R}$,

$$\big(F(t)\big)\, x = \begin{cases} x \displaystyle\sum_{k=1}^{n} y_k & \text{if} \;\; t \in (\frac{1}{n+1}, \frac{1}{n}], \;\; n \in \mathbb{N}, \\ \;\; x\, S & \text{if} \;\; t = 0 \end{cases}$$

where $y_k = \frac{1}{k} e_k \in \ell_2$ [1]for $k \in \mathbb{N}$, and $S = \sum_{k=1}^{\infty} y_k$.

[1]For $k \in \mathbb{N}$, e_k denotes an element of the canonical Schauder basis of ℓ_2.

We will prove that

$$\mathrm{SV}_0^1(F) < \mathrm{SV}_0^{\frac{1}{2}}(F) + \mathrm{SV}_{\frac{1}{2}}^1(F). \tag{2.6}$$

First, let us calculate $\mathrm{SV}_0^{\frac{1}{2}}(F)$.

Given a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[0, \frac{1}{2}]$, as in Example 1.3, put

$$n_j = \max\{k \in \mathbb{N} : k\alpha_j \leq 1\} \quad \text{for} \quad j = 1, \dots, \nu(D),$$

and $\Lambda = \{j : n_j < n_{j-1}\} \subset \{2, \dots, \nu(D)\}$. For $x_j \in \mathbb{R}$, $j = 1, \dots, \nu(D)$ with $|x_j| \leq 1$ we have

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_2 = \Big\| \sum_{k=1}^{\infty} \lambda_k\, y_k \Big\|_2 = \Big(\sum_{k=3}^{\infty} \Big| \frac{\lambda_k}{k} \Big|^2 \Big)^{1/2} \leq \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2}.$$

where λ_k for $k \in \mathbb{N}$, $k \geq 3$, is given as in (1.1) (note that the corresponding n_j satisfies $n_j \geq 2$, $j = 1, \dots, \nu(D)$). In view of this, it is clear that

$$\mathrm{SV}_0^{\frac{1}{2}}(F) \leq \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2} = \sqrt{\frac{\pi^2}{6} - \frac{5}{4}}.$$

The equality $\mathrm{SV}_0^{\frac{1}{2}}(F) = \sqrt{\frac{\pi^2}{6} - \frac{5}{4}}$ is a consequence of the fact that

$$V(F, D_N, [0, \frac{1}{2}]) = \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2}$$

for any division $D_N = \Big\{0, \frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}\Big\}$ with $N \in \mathbb{N}$.

On the other hand, it is not hard to see that $\mathrm{SV}_{\frac{1}{2}}^1(F) = \frac{1}{2}$. Indeed, for any division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[\frac{1}{2}, 1]$ and for any choice of $x_j \in \mathbb{R}$, $j = 1, \dots, \nu(D)$ with $|x_j| \leq 1$ we have

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j = [F(\alpha_1) - F(\frac{1}{2})]\, x_1 = -x_1\, y_2 = -\frac{x_1}{2}\, e_2$$

Hence, $V(F, D, [\frac{1}{2}, 1]) = \frac{1}{2}$.

Recalling that $\mathrm{SV}_0^1(F) = \sqrt{\frac{\pi^2}{6} - 1}$, we conclude that (2.6) holds.

In the sequel we provide some further characterizations of the semivariation of a function. The first one, Theorem 2.10, can be found for instance in [34, Proposition 1.1] or [22, Theorem I.4.4]. Basically, it connects the notions of semivariation and $\mathcal{B}^*$-variation, with respect to the bilinear triple $\mathcal{B}^* = (L(X,Y), L(X), L(X,Y))$ (for definition see [22]).

Theorem 2.10. *For* $F : [a,b] \to L(X,Y)$ *and* $D \in \mathcal{D}[a,b]$ *put*

$$V^*(F,D) = \sup\left\{ \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} \,:\, G_j \in L(X),\ \|G_j\|_{L(X)} \le 1 \right\}.$$

Then

$$\mathrm{SV}_a^b(F) = \sup\{V^*(F,D) \,:\, D \in \mathcal{D}[a,b]\}.$$

Proof. It is enough to show that

$$V^*(F,D) = V(F,D) \quad \text{for every} \quad D \in \mathcal{D}[a,b].$$

Let $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ be a division of $[a,b]$. For $G_j \in L(X)$, $j = 1, \dots, \nu(D)$ with $\|G_j\|_{L(X)} \le 1$ we have

$$\begin{aligned}
\Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} &= \sup_{\|z\|_X \le 1} \Big\| \Big(\sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big) z \Big\|_Y \\
&= \sup_{\|z\|_X \le 1} \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, (G_j\, z) \Big\|_Y \\
&\le \sup_{\|y_j\|_X \le 1} \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, y_j \Big\|_Y
\end{aligned}$$

(where the last inequality is due to the fact that $\|G_j\, z\|_X \le 1$ provided $\|z\|_X \le 1$). Hence $V^*(F,D) \le V(F,D)$.

To obtain the reversed inequality, let us choose $w \in X$ and $\varphi \in X^*$ such that $\|w\|_X = 1$, $\|\varphi\|_{X^*} = 1$ and $\varphi(w) = 1$ (which exists by the Hahn-Banach theorem, c.f. [19, Theorem 2.7.4]). Given $x_j \in X$, $j = 1, \dots, \nu(D)$ with $\|x_j\|_X \le 1$ consider $G_j \in L(X)$ defined by

$$G_j\, x = \varphi(x)\, x_j \quad \text{for} \quad x \in X.$$

Note that, $\|G_j\|_{L(X)} \le 1$ and $G_j\, w = x_j$ for $j = 1, \dots, \nu(D)$. Thus

$$\begin{aligned}
\Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, x_j \Big\|_Y &= \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, (G_j\, w) \Big\|_Y \\
&= \Big\| \Big(\sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big) w \Big\|_Y \\
&\le \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} \|w\|_X
\end{aligned}$$

which yields $V(F,D) \le V^*(F,D)$ concluding the proof. $\square$

Remark 2.11. In a more general formulation, $V^*(F,D)$ in Theorem 2.10 can be defined so that the supremum is taken over all possible choices of $G_j \in L(Z,X)$ with $\|G_j\|_{L(Z,X)} \le 1$, $j = 1,\dots,\nu(D)$; where Z is an arbitrary Banach space.

The following theorem, stated in [22, 3.6, Chapter I], will be useful for the investigation of continuity type results for semivariation. The characterization presented involves functions $(y^* \circ F) : [a,b\,] \to X^*$, obtained by the composition of $F : [a,b\,] \to L(X,Y)$ and a functional $y^* \in Y^*$ which reads as follows

$$(y^* \circ F)(t)(x) = y^*\big(F(t)\,x\big) \quad \text{for} \;\; t \in [a,b\,],\; x \in X. \tag{2.7}$$

Theorem 2.12. *The semivariation of a function $F : [a,b\,] \to L(X,Y)$ is given by*

$$\mathrm{SV}_a^b(F) = \sup\big\{\, \mathrm{var}_a^b\,(y^* \circ F) \;:\; y^* \in Y^*,\; \|y^*\|_{Y^*} \le 1\big\}. \tag{2.8}$$

Moreover, $F \in SV([a,b\,], L(X,Y))$ if and only if $(y^* \circ F) \in BV([a,b\,], X^*)$ for all $y^* \in Y^*$.

Proof. Let $D \in \mathcal{D}[a,b\,]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$, be given. For $x_j \in X$, $j = 1,\dots,\nu(D)$ with $\|x_j\|_X \le 1$ we have

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\,x_j \Big\|_Y \\
&\quad = \sup\left\{ \Big| y^*\Big(\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\,x_j \Big) \Big| \;:\; y^* \in Y^*,\; \|y^*\|_{Y^*} \le 1 \right\} \\
&\quad \le \sup\left\{ \sum_{j=1}^{\nu(D)} \big| [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\,x_j \big| \;:\; y^* \in Y^*,\; \|y^*\|_{Y^*} \le 1 \right\}.
\end{aligned}$$

Therefore,

$$V(F,D) \le \sup\left\{ \sum_{j=1}^{\nu(D)} \|y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})\|_{X^*} \;:\; y^* \in Y^*,\; \|y^*\|_{Y^*} \le 1 \right\}$$

for every $D \in \mathcal{D}[a,b\,]$, and consequently

$$\mathrm{SV}_a^b(F) \le \sup\big\{\, \mathrm{var}_a^b\,(y^* \circ F) \;:\; y^* \in Y^*,\; \|y^*\|_{Y^*} \le 1\big\}. \tag{2.9}$$

On the other hand, given $y^* \in Y^*$ with $\|y^*\|_{Y^*} \le 1$ and $\varepsilon > 0$, for $j = 1,\dots,\nu(D)$, there exists $x_j \in X$ with $\|x_j\|_X \le 1$ such that

$$\|y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})\|_{X^*} - \frac{\varepsilon}{\nu(D)} < \big| [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\,x_j \big|.$$

If we put $\lambda_j := \operatorname{sgn}\big([y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, x_j\big)$ and $\tilde{x}_j = \lambda_j\, x_j$ for $j = 1, \dots, \nu(D)$, then we obtain

$$
\begin{aligned}
&\sum_{j=1}^{\nu(D)} \|y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})\|_{X^*} - \varepsilon \\
&\qquad \le \sum_{j=1}^{\nu(D)} [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, \tilde{x}_j \\
&\qquad \le \Big| \sum_{j=1}^{\nu(D)} [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, \tilde{x}_j \Big| = \Big| y^* \Big(\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, \tilde{x}_j \Big) \Big| \\
&\qquad \le \|y^*\|_{Y^*} \Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, \tilde{x}_j \Big\|_Y \le \mathrm{SV}_a^b(F)
\end{aligned}
$$

Taking the surpremum over all $D \in \mathcal{D}[a,b\,]$, we get $\operatorname{var}_a^b(y^* \circ F) \le \varepsilon + \mathrm{SV}_a^b(F)$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$
\operatorname{var}_a^b(y^* \circ F) \le \mathrm{SV}_a^b(F) \quad \text{for } y^* \in Y^* \text{ with } \|y^*\|_{Y^*} \le 1,
$$

which, together with (2.9), proves the result. $\square$

The equality in (2.8) is used, in a more general way, to define the notion of semivariation in the frame of functions with values in an arbitrary Banach space (cf. [4]). More precisely, if Z is a Banach space, the semivariation of $f : [a,b\,] \to Z$ is given by

$$
\operatorname{semivar}_a^b(f) = \sup \Big\{ \operatorname{var}_a^b (z^* \circ f) \,:\, z^* \in Z^*,\ \|z^*\|_{z^*} \le 1 \Big\}.
$$

where the functions $(z^* \circ f) : [a,b\,] \to \mathbb{R}$ are defined as in (2.7) with an obvious adaptation.

Thereafter, for operator-valued functions two notions of semivariation can be derived. However, no direct connection between them is established since such connection would rely on a characterization of the dual space of $L(X,Y)$. On the other hand, as observed in [4], given a function $F : [a,b\,] \to L(X,Y)$, those two notions are related as follows: for each $x \in X$ the function $F_x : t \in [a,b\,] \mapsto F(t)x \in Y$ satisfies

$$
\operatorname{semivar}_a^b(F_x) \le \mathrm{SV}_a^b(F).
$$

3 Semivariation and variation

We have mentioned in Section 1 that every function of bounded variation is also of bounded semivariation, that is,

$$
BV([a,b\,], L(X,Y)) \subseteq SV([a,b\,], L(X,Y)). \tag{3.1}
$$

This section is devoted to the study of conditions ensuring the equality of these two sets. To start, we investigate the case when Y is the real line.

Theorem 3.1. *Let $F : [a,b] \to X^*$ be given. Then, $F \in SV([a,b],X^*)$ if and only if $F \in BV([a,b],X^*)$. In this case, $\mathrm{SV}_a^b(F) = \mathrm{var}_a^b(F)$.*

Proof. is analogous to the proof of Theorem 2.12. In summary, it is a consequence of the fact that we can write

$$\begin{aligned} V(F,D) &= \sup\left\{ \Big|\sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, x_j\Big| \,:\, x_j \in X,\ \|x_j\|_X \le 1 \right\} \\ &= \sum_{j=1}^{\nu(D)} \big\|F(\alpha_j) - F(\alpha_{j-1})\big\|_{X*} \end{aligned}$$

for every division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ of $[a,b]$. □

Remark 3.2. Given $n \in \mathbb{N}$ consider a function $F : [a,b] \to L(X,\mathbb{R}^n)$. Writing $F = (F_1, \ldots, F_n)$ with $F_j : [a,b] \to X^*$, $j = 1, \ldots, n$, it is clear that

$$F \in BV([a,b], L(X,\mathbb{R}^n)) \text{ if and only if } F_j \in BV([a,b],X^*),\ j = 1,\ldots,n$$

and similarly

$$F \in SV([a,b], L(X,\mathbb{R}^n)) \text{ if and only if } F_j \in SV([a,b],X^*),\ j = 1,\ldots,n.$$

With this in mind, the assertion in Theorem 3.1 can be extended to the case when Y is an Euclidean space. More generally: *if Y is a finite dimensional Banach space, we have*

$$F \in SV([a,b], L(X,Y)) \textit{ if and only if } F \in BV([a,b], L(X,Y)).$$

The following example presents a function of bounded semivariation whose variation is not finite.

Example 3.3. Let $F : [0,1] \to L(\mathbb{R},\ell_2)$ be the function given on Example 1.3, that is, for $t \in [0,1]$ and $x \in \mathbb{R}$,

$$\big(F(t)\big)\,x = \begin{cases} x\displaystyle\sum_{k=1}^{n} y_k & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}],\ n \in \mathbb{N}, \\ x\,S & \text{if } t = 0 \end{cases} \tag{3.2}$$

where $y_k = \frac{1}{k} e_k \in \ell_2$ [2]

[2]For $k \in \mathbb{N}$, e_k denotes an element of the canonical Schauder basis of ℓ_2.

We know that $F \in SV([0,1], L(\mathbb{R}, \ell_2))$. On the other hand, since

$$\|F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})\|_{L(\mathbb{R},\ell_2)} = \|y_{k+1}\|_2 = \frac{1}{k+1} \quad \text{for every} \quad k \in \mathbb{N},$$

we have

$$\sum_{k=1}^{N+1} \frac{1}{k} \leq \sum_{k=1}^{N} \|F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})\|_{L(\mathbb{R},\ell_2)} + \|F(\tfrac{1}{N+1}) - F(0)\|_{L(\mathbb{R},\ell_2)} \leq \operatorname{var}_0^1(F),$$

for any choice of $N \in \mathbb{N}$. Therefore $\operatorname{var}_0^1(F) = \infty$.

The main tool for the construction of the function in the example above was the sequence $\{y_n\}_n$ in ℓ_2 whose series converges but not absolutely. Recalling that for infinite dimensional Banach spaces we can always find a sequence with such property (due to Dvoretzky-Rogers Theorem A.5 presented in the appendix), one can see that finite dimension is a necessary and sufficient condition for the equivalence between bounded variation and bounded semivariation. We remark that in [40, Theorem 2] this equivalence was actually proved for functions defined on a ring of sets.

Using the ideas from [40], we will show that for infinite dimensional spaces Y the inclusion in (3.1) is strict.

Theorem 3.4. *If the dimension of Y is infinite, then there exists $F \in SV([a,b], L(X,Y))$ such thar $\operatorname{var}_a^b(F) = \infty$.*

Proof. By the Dvoretzky-Rogers Theorem A.5 and its Corollary A.6 in the Appendix, there exists a sequence $\{y_n\}_n$ in Y such that the series $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent but not absolutely convergent. Considering an increasing sequence $\{t_n\}_n$ in (a,b) converging to b and fixing an arbitrary $\varphi \in X^*$, with $\|\varphi\|_{X^*} = 1$, let

$$F(t)\,x = \sum_{t_k < t} \varphi(x)\, y_k \quad \text{for} \quad x \in X \quad \text{and} \quad t \in [a,b].$$

Note that $F : [a,b] \to L(X,Y)$ is well-defined (see Theorem A.3 in the Appendix).

We claim that the variation of F is not finite. Indeed, given $N \in \mathbb{N}$ consider the division $D_N = \{t_0, t_1, \ldots, t_{N+1}, b\}$ formed by elements of the sequence $\{t_n\}_n$ and $t_0 = a$. Noting that

$$\|y_k\|_Y = \|F(t_{k+1}) - F(t_k)\|_{L(X,Y)} \quad \text{for every} \quad k \in \mathbb{N},$$

we have

$$\sum_{k=1}^{N} \|y_k\|_Y \leq \sum_{j=1}^{N+1} \|F(t_j) - F(t_{j-1})\|_{L(X,Y)} + \|F(b) - F(t_{N+1})\|_{L(X,Y)} \leq \operatorname{var}_a^b(F).$$

Since $\sum_{n=1}^{\infty} y_n$ is not absolutely convergent, it follows that $\mathrm{var}_a^b(F) = \infty$.

Let us show that $F \in SV([a,b], L(X,Y))$. Consider a division $D = \{\alpha_0, \ldots, \alpha_{\nu(D)}\}$ of $[a,b]$ and let $x_j \in X$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_X \leq 1$. Thus

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_Y = \Big\| \sum_{j=1}^{\nu(D)} \big(\varphi(x_j) \sum_{t_k \in [\alpha_{j-1}, \alpha_j)} y_k\big) \Big\|_Y = \Big\| \sum_{k=1}^{\infty} \beta_k y_k \Big\|_Y$$

where $\beta_k = \varphi(x_j)$ if $t_k \in [\alpha_{j-1}, \alpha_j)$, for some $j = 1, \ldots, \nu(D)$, otherwise $\beta_k = 0$. By Lemma A.4 from the Appendix we know that the set

$$\left\{ \sum_{n=1}^{\infty} \lambda_n\, y_n \;:\; \lambda_n \in \mathbb{R} \;\text{ with }\; |\lambda_n| \leq 1,\; n \in \mathbb{N} \right\}$$

is bounded in Y. Thus, $\Big\| \sum_{k=1}^{\infty} \beta_k y_k \Big\|_Y$ is bounded (uniformly with respect to the choice of $x_j \in X$, $j = 1, \ldots, \nu(D)$) and, consequently, $\mathrm{SV}_a^b(F) < \infty$ which proves the result. $\square$

According to Remark 3.2 and Theorem 3.4 we conclude that the notion of semivariation is relevant only in spaces with infinite dimension.

Corollary 3.5. *The following assertions are equivalent:*

(*i*) *the dimension of* Y *is finite;*

(*ii*) *every function* $F \in SV([a,b], L(X,Y))$ *is of bounded variation on* $[a,b]$.

Regarding the function F in (3.2), it was shown on Example 2.9 that its semivariation is not additive with respect to intervals (see (2.6)). It turns out that such additivity type property can be used to identify whether a function of bounded semivariation has a bounded variation as well. This is the content of the following theorem.

Theorem 3.6. *Let* $F \in SV([a,b], L(X,Y))$. *Then* $F \in BV([a,b], L(X,Y))$ *if and only if*

$$M := \sup \left\{ \sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \;:\; D \in \mathcal{D}[a,b] \right\} < \infty. \tag{3.3}$$

Moreover, in this case, $\mathrm{var}_a^b(F) = M$.

Proof. Assume (3.3) holds. Given $\varepsilon > 0$, for $D \in \mathcal{D}[a,b]$, with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$, we can choose $x_j \in X$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_X \leq 1$ such that

$$\|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} - \frac{\varepsilon}{\nu(D)} < \|\, [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \|_Y.$$

Noting that, for $j = 1, \dots, \nu(D)$,

$$\| [F(\alpha_j) - F(\alpha_{j-1})] \, x_j \|_Y \le \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F),$$

it follows that

$$\sum_{j=1}^{\nu(D)} \|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} - \varepsilon < \sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \le M.$$

Therefore, taking the supremum over all divisions $D \in \mathcal{D}[a, b\,]$ we obtain

$$\mathrm{var}_a^b(F) < M + \varepsilon.$$

Consequently $F \in BV([a, b\,], L(X, Y))$ and, since $\varepsilon > 0$ is arbitrary, $\mathrm{var}_a^b(F) \le M$.

On the other hand, for any division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[a, b\,]$ we have

$$\sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \le \sum_{j=1}^{\nu(D)} \mathrm{var}_{\alpha_{j-1}}^{\alpha_j}(F) = \mathrm{var}_a^b(F),$$

wherefrom we conclude that $\mathrm{var}_a^b(F) = M$. $\square$

4 Semivariation: limits and continuity

It is well-known that a function of bounded variation is regulated, that is, the one-sided limits exist at every point of the domain (see [20, Theorem I.2.7] or [14, Lemma 2.1]). In this section we investigate the connection between functions of bounded semivariation and regulated functions.

Following the notation in [22], if $f : [a, b\,] \to X$ is a regulated function $[a, b\,]$, we write $f \in G([a, b\,], X)$, and the one-sided limits are denoted by

$$f(t-) = \lim_{s \to t-} f(s) \quad \text{and} \quad f(t+) = \lim_{s \to t+} f(s)$$

for $t \in [a, b\,]$ with the convention $f(a-) = f(a)$ and $f(b+) = f(b)$.

Another useful notion through this section is the semivariation on half-closed intervals.

Definition 4.1. Given $F : [a, b\,] \to L(X, Y)$ and $c, d \in [a, b\,]$, $c < d$, the semivariation of F on a half-closed interval $[c, d)$ is given by

$$\mathrm{SV}_{[c,d)}(F) = \lim_{t \to d-} \mathrm{SV}_c^t(F) = \sup_{t \in [c,d)} \mathrm{SV}_c^t(F).$$

In analogous way, we define the semivariation on the half-closed interval $(c, d]$ by

$$\mathrm{SV}_{(c,d]}(F) = \lim_{t \to c+} \mathrm{SV}_t^d(F) = \sup_{t \in (c,d]} \mathrm{SV}_t^d(F).$$

Theorems 2.7 and 2.8 guarantee that the semivariation over half-closed subintervals of $[a,b]$ is finite for every function from $SV([a,b],L(X,Y))$.

In what follows we show that a function of bounded semivariation is regulated provided some conditions on the semivariation over half-closed intervals are satisfied.

Theorem 4.2. *Let $F \in SV([a,b],L(X,Y))$ be such that*

$$\lim_{\delta\to 0+} \mathrm{SV}_{[t-\delta,t)}(F) = 0 \quad \textit{for every} \;\; t \in (a,b], \tag{4.1a}$$

$$\lim_{\delta\to 0+} \mathrm{SV}_{(t,t+\delta]}(F) = 0 \quad \textit{for every} \;\; t \in [a,b). \tag{4.1b}$$

Then F is a regulated function on $[a,b]$.

Proof. Given $t \in (a,b]$ we will prove that $F(t-) \in L(X,Y)$ exists. To this aim, consider an increasing sequence $\{t_n\}_n$ in (a,t) converging to t.

Let $\varepsilon > 0$ be given. By (4.1a) there exists $\delta > 0$ such that

$$\mathrm{SV}_{[t-\delta,t)}(F) < \varepsilon$$

Moreover, there is $N \in \mathbb{N}$ so that $t_n > t-\delta$ for every $n \geq N$. Thus, for $m > n > N$ and $x \in X$ with $\|x\|_X \leq 1$ we obtain

$$\|[F(t_m) - F(t_n)]x\|_Y \leq \mathrm{SV}_{t-\delta}^{t_m}(F) \leq \mathrm{SV}_{[t-\delta,t)}(F) < \varepsilon$$

which implies that $F(t-)$ exists. Analogously, using (4.1b), we can show the existence of $F(t+)$ for every $t \in [a,b)$. $\square$

Remark 4.3. It is not hard to see that, replacing (4.1a) and (4.1b) by

$$\lim_{\delta\to 0+} \mathrm{SV}_{t-\delta}^{t}(F) = 0 \quad \text{and} \quad \lim_{\delta\to 0+} \mathrm{SV}_{t}^{t+\delta}(F) = 0 \;\; \text{for every} \;\; t \in [a,b],$$

it follows that F is continuous on $[a,b]$.

Next lemma is the analogue of [20, Proposition 4.13] and provides a condition ensuring that (4.1a) and (4.1b) hold.

Lemma 4.4. *If Y is a weakly sequentially complete Banach space, then* (4.1a) *and* (4.1b) *are satisfied for every $F \in SV([a,b],L(X,Y))$.*

Proof. By contradiction assume that there exists a function $F \in SV([a,b],L(X,Y))$ such that for some $t \in (a,b]$ we have $\lim_{\delta\to 0+} \mathrm{SV}_{[t-\delta,t)}(F) = M > 0$. Hence, there is $\delta_1 > 0$ such that

$$\sup_{s\in[t-\delta,t)} \mathrm{SV}_{t-\delta}^{s}(F) = \mathrm{SV}_{[t-\delta,t)}(F) > \frac{M}{2} \quad \text{for} \;\; 0 < \delta \leq \delta_1.$$

Put $s_1 = t - \delta_1$. In view of the inequality above, there exists $s_2 \in (s_1, t)$ so that

$$\mathrm{SV}_{s_1}^{s_2}(F) > \frac{M}{2}.$$

Moreover, $\mathrm{SV}_{[s_2,t)}(F) > \frac{M}{2}$. Thus, we can choose $s_3 \in (s_2, t)$ with

$$\mathrm{SV}_{s_2}^{s_3}(F) > \frac{M}{2} \quad \text{and} \quad \mathrm{SV}_{[s_3,t)}(F) > \frac{M}{2}.$$

If we proceed in this way, we obtain an increasing sequence $\{s_n\}_n$ in (a, t) such that

$$\lim_{n\to\infty} s_n = t \quad \text{and} \quad \mathrm{SV}_{s_n}^{s_{n+1}}(F) > \frac{M}{2}, \quad n \in \mathbb{N}.$$

Having this in mind, for each $n \in \mathbb{N}$, we can find a division $D_n = \{\alpha_0^{(n)}, \alpha_1^{(n)}, \dots, \alpha_{\nu_n}^{(n)}\}$ of $[s_n, s_{n+1}]$ and $x_j^{(n)} \in X$, $j = 1, \dots, \nu_n$ with $\|x_j^{(n)}\|_X \le 1$ such that

$$\Big\| \sum_{j=1}^{\nu_n} [F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})] x_j^{(n)} \Big\|_Y > \frac{M}{2}$$

Let

$$y_n = \sum_{j=1}^{\nu_n} [F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})] x_j^{(n)} \quad \text{for} \ \ n \in \mathbb{N}.$$

We claim that $\sum_{n=1}^{\infty} |y^*(y_n)| < \infty$ for every $y^* \in Y^*$ with $\|y^*\|_{Y^*} \le 1$. Indeed, given $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=1}^{N} |y^*(y_n)| &= \sum_{n=1}^{N} \Big| \sum_{j=1}^{\nu_n} y^*\Big([F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})] x_j^{(n)}\Big) \Big| \\
&\le \sum_{n=1}^{N} \sum_{j=1}^{\nu_n} \|y^* \circ F(\alpha_j^{(n)}) - y^* \circ F(\alpha_{j-1}^{(n)})\|_{X^*} \\
&\le \sum_{n=1}^{N} \mathrm{var}_{s_n}^{s_{n+1}}(y^* \circ F) = \mathrm{var}_{s_1}^{s_{N+1}}(y^* \circ F).
\end{aligned}$$

which together with Theorem 2.12 leads to

$$\sum_{n=1}^{N} |y^*(y_n)| \le \mathrm{SV}_{s_1}^{s_{N+1}}(F) \le \mathrm{SV}_{[s_1,t)}(F) < \infty.$$

Thus, we conclude that the series $\sum_{n=1}^{\infty} y_n$ is weakly (unconditionally) convergent. Since Y is weakly sequentially complete, it follows that $\sum_{n=1}^{\infty} y_n$ converges in Y (see Theorem A.10 in the Appendix). This contradicts the fact that $\|y_n\|_Y > \frac{M}{2} > 0$ for every $n \in \mathbb{N}$.

In summary, we conclude that (4.1a) holds for every function from $SV([a,b], L(X,Y))$. Analogously we can show that (4.1b) is also true. $\square$

The results above, together with [22, Corollary I.3.2], lead to the following conclusion about the continuity of a function of bounded semivariation.

Corollary 4.5. *Let $F \in SV([a,b], L(X,Y))$. If Y is a weakly sequentially complete Banach space, then F is continuous on $[a,b]$ except for a countable set.*

Remark 4.6. Recalling that reflexive spaces are weakly sequentially complete (see [19, Theorem 2.10.3]), Lemma 4.4, as well as Corollary 4.5, remains valid for Y reflexive.

According to the characterization given in Theorem 2.12, for $F \in SV([a,b], L(X,Y))$ and $y^* \in Y^*$, the function $y^* \circ F : [a,b] \to X^*$ is of bounded variation on $[a,b]$. This implies that for each $t \in [a,b]$ both limits

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) \quad \text{and} \quad \lim_{\delta \to 0+} y^* \circ F(t+\delta)$$

exist in X^*. Such limits can described by means of an operator mapping X into the second dual Y^{**} of Y. Now, we need to fix some notation to make our statement more precise.

Given $U \in L(X, Y^{**})$ and $y^* \in Y^*$, we can define a linear functional $y^* \bullet U : X \to \mathbb{R}$ by setting $(y^* \bullet U)(x) = \big(U(x)\big)(y^*)$ for every $x \in X$.

Theorem 4.7. *Let $F \in SV([a,b], L(X,Y)$. Then, for each $t \in (a,b]$ and $s \in [a,b)$, there exist $F(t-)$, $F(s+) \in L(X,Y^{**})$ such that, for every $y^* \in Y^*$,*

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) = y^* \bullet F(t-) \quad \text{and} \quad \lim_{\delta \to 0+} y^* \circ F(s+\delta) = y^* \bullet F(s+)$$

where $y^* \circ F$ is as in (2.7).

Proof. Without loss of generality, let us assume $F(a) = 0$. Given $t \in (a,b]$, for each $y^* \in Y^*$ there exists $T_{y^*} \in X^*$ such that

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) = T_{y^*}.$$

Considering $T : Y^* \to X^*$ defined by $T(y^*) = T_{y^*}$, $y^* \in Y^*$, clearly T is linear. Moreover, by Theorem 2.12,

$$\|y^* \circ F(t-\delta)\|_{X^*} \le \|y^*\|_{Y^*} \, \mathrm{SV}_a^b(F) \quad \text{for every} \quad y^* \in Y^*,$$

hence $T \in L(Y^*, X^*)$ with $\|T\|_{L(Y^*,X^*)} \le \mathrm{SV}_a^b(F)$.

Let $T^\times : X \to Y^{**}$ be the mapping which associates to each $x \in X$ the linear functional $x^\times : Y^* \to \mathbb{R}$ given by $x^\times(y^*) = T_{y^*}x$ for $y^* \in Y^*$. Note that, for every $x \in X$ and $y^* \in Y^*$, we have

$$\lim_{\delta \to 0+} (y^* \circ F(t-\delta))x = T_{y^*}x = x^\times(y^*) = (y^* \bullet T^\times)(x).$$

Therefore $F(t-) = T^\times \in L(X, Y^{**})$ is the desired operator. Similarly, we can construct $F(s+) \in L(X, Y^{**})$ for $s \in [a,b)$. $\square$

The theorem above suggests that functions of bounded semivariation are regulated in some weak sense. For operator-valued functions a more general notion of regulated function can be defined.

Definition 4.8. Given $F : [a, b\,] \to L(X, Y)$, we say F is simply regulated on $[a, b\,]$ if, for each $x \in X$, the function $t \in [a, b\,] \longmapsto F(t)\, x \in Y$ is regulated. We will denote the set of such functions by $SG([a, b\,], L(X, Y))$.

From the Banach-Steinhaus Theorem (c.f. [19, Theorem 2.11.4]), given a function $F \in SG([a, b\,], L(X, Y))$, for each $t \in (a, b]$ there exists $F(t\dot{-}) \in L(X, Y)$ such that

$$\lim_{s \to t-} F(s)x = F(t\dot{-})x \quad \text{for every} \quad x \in X.$$

Analogously, for $t \in [a, b)$, we have $F(t\dot{+}) \in L(X, Y)$ satisfying $\lim\limits_{s \to t+} F(s)x = F(t\dot{+})x$ for every $x \in X$.

The concept of simply regulated function appears in the literature under different nomeclatures (see [22] and [33]), for instance, weakly regulated or $(\mathcal{B})$-regulated with respect to the bilinear triple $\mathcal{B} = (L(X, Y), X, Y)$. Our choice follows the work of Honig in [23], among other of his publications and followers (see also [3]). In some sense, such terminology could be seen as reference to the notion of regulated function in the weak* topology - also known as simple topology.

By the Definition 4.8, it is clear

$$G([a, b\,], L(X, Y)) \subset SG([a, b\,], L(X, Y))$$

(for details, see [33, Proposition 3])

Recalling that $BV([a, b\,], X) \subset G([a, b\,], X)$, we could expect that a similar relation would hold in the frame of functions of bounded semivarition relatively to the notion of simply regulated functions defined above. The following example, inspired by [4], shows that this is not the case.

Example 4.9. Let ℓ_∞ be the Banach space of bounded sequences $x = \{x_n\}_n$ in $\mathbb{R}$, endowed with the usual supremum norm

$$\|x\|_\infty = \sup\{\, |x_n| : n \in \mathbb{N}\}.$$

Denote by e_k, $k \in \mathbb{N}$, the canonical basis of ℓ_∞, where e_k is the sequence which is 1 in the k-th coordinate and null elsewhere.

Consider the function $F : [0, 1] \to L(\ell_\infty)$ given by

$$\big(F(t)\big)\, x = \begin{cases} x_1\, e_n & \text{if} \;\; t \in (\frac{1}{n+1}, \frac{1}{n}],\;\; n \in \mathbb{N}, \\[2mm] 0 & \text{if} \;\; t = 0 \end{cases}$$

for $t \in [0,1]$ and $x = \{x_n\}_n \in \ell_\infty$.

Note that, for every $k \in \mathbb{N}$,

$$\|[F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})]\, e_1\|_\infty = \|e_k - e_{k+1}\|_\infty = 1.$$

Hence $\lim_{k\to\infty} \big(F(\frac{1}{k})\big)e_1$ does not exist and, consequently, neither do $F(0+)$. This shows that F is not simply regulated.

Let us prove that $F \in SV([0,1], L(\ell_\infty))$. Given a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ of $[0,1]$, let $k_j = \max\{k \in \mathbb{N}\,;\, k\alpha_j \le 1\}$ for $j = 1, \ldots, \nu(D)$. Considering $x_j \in \ell_\infty$, $x_j = \{x_n^{(j)}\}_n$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_\infty \le 1$, we have

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j = x_1^{(1)}\, e_{k_1} + \sum_{j=2}^{\nu(D)} \big[x_1^{(j)}\, e_{k_j} - x_1^{(j)}\, e_{k_{j-1}}\big]$$

Taking $\Lambda = \{j \,:\, k_j \ne k_{j-1}\} \subset \{2, \ldots, \nu(D)-1\}$, we can write

$$\begin{aligned}\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j &= x_1^{(1)}\, e_{k_1} + \sum_{j\in\Lambda} \big[x_1^{(j)}\, e_{k_j} - x_1^{(j)}\, e_{k_{j-1}}\big] + x_1^{(\nu(D))}\, e_{k_{\nu(D)}} \\ &= \sum_{j\in\Lambda\cup\{1\}} \lambda_j\, e_{k_j} + x_1^{(\nu(D))}\, e_{k_{\nu(D)}}\end{aligned}$$

where, for each $j \in \Lambda \cup \{1\}$, λ_j corresponds to the difference between two elements of the set $\{x_1^{(i)} \,:\, i = 1, \ldots, \nu(D)-1\}$. Clearly $|\lambda_j| \le 2$, thus

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j \Big\|_\infty \le 2$$

which implies that $\mathrm{SV}_0^1(F) < \infty$.

In view of this, a quite natural question arises: under which conditions is the space $SV([a,b], L(X))$ contained in the set of simply regulated functions?

In [3, Theorem 1] it was proved that the inclusion holds whenever X is a uniformly convex Banach space. Later, a final answer was given in [4], where a necessary and sufficient condition was established.

Aiming to present such result here, we have to consider a very special class of spaces, namely, all Banach spaces which do not contain an isomorphic copy of c_0 (by c_0 we denote the space of sequences in $\mathbb{R}$ converging to zero with respect to the supremum norm). By the Theorem of Bessaga and Pelczinsky (see Theorem A.9 in the Appendix), the fact that a Banach space X does not contain a copy of c_0 is equivalent to the following property:

(BP) *all series* $\sum x_n$ *in* X *such that* $\sum |x^*(x_n)| < \infty$ *for every* $x^* \in X^*$ *are unconditionally convergent.*

Using this caractherization, we will present in details the relation between the sets $SV([a,b], L(X))$ and $SG([a,b], L(X))$ described in [4, Theorem 5].

Theorem 4.10. *The following assertions are equivalent:*

(1) X *does not contain an isomorphic copy of* c_0

(2) *every function* $F : [a,b] \to L(X)$ *of bounded semivariation is simply regulated.*

The proof of this theorem is contained in the following two lemmas.

Lemma 4.11. *If* X *does not contain an isomorphic copy of* c_0*, then*

$$SV([a,b], L(X)) \subset SG([a,b], L(X)).$$

Proof. Given $F \in SV([a,b], L(X))$ and $x \in X$, $x \neq 0$, let $F_x : [a,b] \to X$ be the function given by

$$F_x(t) = F(t)\, x \quad \text{for} \;\; t \in [a,b].$$

Fixed an arbitrary $t \in (a,b]$, to show that the left-sided limit $F_x(t-)$ exists, consider an increasing sequence $\{t_n\}_n$ on (a,t) converging to t.

Let $x^* \in X^*$. For $N \in \mathbb{N}$, taking the division $D_N = \{t_0, t_1, \ldots, t_N, b\}$ of $[a,b]$ formed by elements of the sequence $\{t_n\}_n$ and $t_0 = a$, we have

$$\sum_{j=1}^{N} |x^*(F_x(t_j) - F_x(t_{j-1}))| = x^*\left(\sum_{j=1}^{N} [F(t_j) - F(t_{j-1})]\, \lambda_j\, x\right)$$

where $\lambda_j = \operatorname{sgn}(x^*(F_x(t_j) - F_x(t_{j-1})))$ for $j = 1, \ldots, N$. If we put $x_j = \frac{\lambda_j\, x}{\|x\|_X}$, we get

$$\begin{aligned}\sum_{j=1}^{N} |x^*\big(F_x(t_j) - F_x(t_{j-1})\big)| &\leq \|x^*\|_{X^*} \|x\|_X \left\| \sum_{j=1}^{N} [F(t_j) - F(t_{j-1})]\, x_j \right\|_X \\ &\leq \|x^*\|_{X^*}\, \|x\|_X\, \mathrm{SV}_a^b(F).\end{aligned}$$

Since the inequality is valid for every $N \in \mathbb{N}$, we conclude that

$$\sum_{n=1}^{\infty} |x^*(F_x(t_n) - F_x(t_{n-1}))| < \infty, \quad x^* \in X^*.$$

By the property (BP) of the space X, the series $\sum_{n=1}^{\infty}\big(F_x(t_n)-F_x(t_{n-1})\big)$ converges to some $z\in X$ and, consequently,

$$\lim_{n\to\infty} F(t_n)\,x=\lim_{n\to\infty}\sum_{k=1}^{n}\big(F_x(t_k)-F_x(t_{k-1})\big)+F_x(a)=z+F(a)\,x.$$

It remains to show that the limit does not depend on the choice of the sequence $\{t_n\}_n$. To this aim, let $\{s_n\}_n$ be another increasing sequence with $\lim_{n\to\infty} s_n=t$. By the same argument used above, there exists $\tilde z\in X$ such that

$$\tilde z=\sum_{n=1}^{\infty}\big(F_x(s_n)-F_x(s_{n-1})\big)\quad\text{and}\quad \lim_{n\to\infty}F_x(s_n)=\tilde z+F(a)\,x.$$

Ordering the set $\{t_n : n\in\mathbb{N}\}\cup\{s_n : n\in\mathbb{N}\}$ we obtain an increasing sequence $\{r_n\}_n$ converging to t whose series $\sum_{n=1}^{\infty}\big(F_x(r_n)-F_x(r_{n-1})\big)$ also converges. Moreover, the limit $\lim_{n\to\infty}F_x(r_n)$ exists. Since $\{F_x(t_n)\}_n$ and $\{F_x(s_n)\}_n$ are convergent subsequences of $\{F_x(r_n)\}_n$, we should have

$$\lim_{n\to\infty}F_x(t_n)=\lim_{n\to\infty}F_x(r_n)=\lim_{n\to\infty}F_x(s_n)$$

which proves that $\tilde z=z$ and $F_x(t-)=z+F(a)\,x$.

Similarly, we can show that the right-sided limit of F_x exists for every $t\in[a,b)$. ☐

The second lemma gives the reverse implication from Theorem 4.10. Roughly speaking, we will show that if c_0 is isomorphically embedded into the space Y, one can construct a function $F:[a,b]\to L(X)$ of bounded semivariation which is not simply regulated.

Lemma 4.12. *If $SV([a,b],L(X))\subset SG([a,b],L(X))$, then X does not contain a copy of c_0.*

Proof. By contradiction, assume that X contains a isomorphic copy of c_0 and denote it by Z. Let $\psi: c_0\to Z$ be an isomorphism and put $z_k:=\psi(e_k)$ where e_k, $k\in\mathbb{N}$, stands for the canonical Schauder basis of c_0.

It is known that there exist positive constants C_1 and C_2 such that, for $N\in\mathbb{N}$, taking $\lambda_j\in\mathbb{R}$, $j=1,\dots,N$, we have

$$C_1\sup_{1\le j\le N}|\lambda_j|\le\Big\|\sum_{j=1}^{N}\lambda_j\,e_j\Big\|_\infty\le C_2\sup_{1\le j\le N}|\lambda_j|,$$

(see [Kadets, Theorem 6.3.1], [Diestel, Theorem V.6]). Thus, by the fact that Z and c_0 are isomorphic,

$$C_1\sup_{1\le j\le N}|\lambda_j|\le\Big\|\sum_{j=1}^{N}\lambda_j\,z_j\Big\|_X\le C_2\sup_{1\le j\le N}|\lambda_j|,\tag{4.2}$$

for $N \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$, $j = 1, \dots, N$.

Using the sequence z_n, $n \in \mathbb{N}$, mentioned above and its properties we will construct a function $F : [a, b] \to L(X)$ in a few steps.

Step 1. Clearly, z_n, $n \in \mathbb{N}$, defines a basis for Z and, for each $k \in \mathbb{N}$, the projection $\pi_k : Z \to \mathbb{R}$, given by $\pi_k(\sum_n \lambda_n z_n) = \lambda_k$, is continuous (see [Diestel, p. 32]). Since (4.2) implies that

$$\left|\pi_k\Big(\sum_{n=1}^{N} z_n\Big)\right| \leq \frac{1}{C_1}\Big\|\sum_{n=1}^{N} z_n\Big\|_Z, \quad N \in \mathbb{N},$$

we have $\|\pi_k\|_{Z^*} \leq \frac{1}{C_1}$ for every $k \in \mathbb{N}$.

Step 2. For $k \in \mathbb{N}$, let $S_k : Z \to Z$ be given by

$$S_k(x) = \sum_{n=1}^{k} \pi_n(x)\, z_{2^k+n}, \quad \text{for} \ \ x \in Z$$

Note that, S_k is a bounded linear operator on Z for every $k \in \mathbb{N}$. Indeed, given $x \in Z$, we can write

$$S_k(x) = \sum_{n=1}^{k} \pi_n(x)\, z_{2^k+n} = \sum_{j=1}^{2^k+k} \beta_j z_j$$

where $\beta_j = \pi_n(x)$ if $j = 2^k + n$ for some $n = 1, \dots, k$, otherwise $\beta_j = 0$. Thus, by (4.2),

$$\|S_k(x)\|_Z \leq C_2 \sup_{1\leq j\leq 2^k+k} |\beta_j| = C_2 \sup_{1\leq n\leq k} |\pi_n(x)| \leq C_2 \sup_{1\leq n\leq k} \|\pi_n\|_{Z^*}\|x\|_Z,$$

which implies that $\|S_k\|_{L(Z)} \leq \frac{C_2}{C_1}$ for every $k \in \mathbb{N}$.

Step 3. Given j, $k \in \mathbb{N}$, put $f_{k,j} = \pi_j \circ S_k$. By the Hahn-Banach theorem, the functional $f_{k,j} \in Z^*$ can be extended to a continuous linear functional $\tilde{f}_{k,j}$ on X satisfying

$$\|\tilde{f}_{k,j}\|_{X^*} = \|f_{k,j}\|_{Z^*} \leq \|\pi_j\|_{Z^*}\|S_k\|_{L(Z)} \leq \frac{C_2}{(C_1)^2}. \tag{4.3}$$

Step 4. For $x \in X$ and $k \in \mathbb{N}$, let $T_k(x) = \sum_{j=1}^{k} \tilde{f}_{k,2^k+j}(x)\, z_{2^k+j}$. Clearly, $T_k \in L(X)$ and it follows from (4.2) and (4.3) that

$$\|T_k(x)\|_X = \Big\|\sum_{j=1}^{k} \tilde{f}_{k,2^k+j}(x)\, z_{2^k+n}\Big\|_X \leq C_2 \sup_{1\leq j\leq k} |\tilde{f}_{k,2^k+j}(x)| \leq \Big(\frac{C_2}{C_1}\Big)^2 \|x\|_X\,, \quad x \in X,$$

that is, $\|T_k\|_{L(X)} \leq \big(\frac{C_2}{C_1}\big)^2$ for all $k \in \mathbb{N}$.

We are now ready to define $F:[a,b\,]\to L(X)$. Given $x\in X$, let

$$F(t)x = T_k(x) \quad \text{for} \;\; t\in(t_{k+1},t_k]$$

where $t_k = a+\frac{(b-a)}{k}$, $k\in\mathbb{N}$.

It is not hard to see that F is not simply regulated. Indeed, for each $k\in\mathbb{N}$, noting that $S_k(z_1)=z_{2^k+1}$, we get

$$F(t_k)z_1 = T_k(z_1) = \sum_{j=1}^{k} f_{k,2^k+j}(z_1)\, z_{2^k+j} = \sum_{j=1}^{k} \pi_{2^k+j}\big(S_k(z_1)\big) z_{2^k+j} = z_{2^k+1},$$

which by (4.2) leads to

$$\|[F(t_k)-F(t_{k+1})]\, z_1\|_X = \|z_{2^k+1}-z_{2^{k+1}+1}\|_X \geq C_1.$$

Hence the limit $\lim_{t\to a+} F(t)\, z_1$ does not exist.

Now, we will show that $F\in SV([a,b\,],L(X))$. Considering a division $D\in\mathcal{D}[a,b\,]$, with $D=\{\alpha_0,\alpha_1,\ldots,\alpha_{\nu(D)}\}$, let $k_j\in\mathbb{N}$ be such that $\alpha_j\in(t_{k_j+1},t_{k_j}]$, $j=1,\ldots,\nu(D)$. For $x_j\in X$, $j=1,\ldots,\nu(D)$ with $\|x_j\|_X\leq 1$ we have

$$\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]x_j = T_{k_1}(x_1) + \sum_{j=2}^{\nu(D)}[T_{k_j}(x_j)-T_{k_{j-1}}(x_j)]$$

Taking $\Lambda=\{j\,:\,k_j\neq k_{j-1}\}\subset\{2,\ldots,\nu(D)-1\}$, we can write

$$\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]x_j = \sum_{j\in\Lambda\cup\{1\}} T_{k_j}(y_j) + T_{k_{\nu(D)}}(x_{\nu(D)})$$

where, for each $j\in\Lambda\cup\{1\}$, $y_j\in X$ corresponds to the difference between two elements of the set $\{x_i\,:\,i=1,\ldots,\nu(D)-1\}$. Noting that $\|y_j\|_X\leq 2$, by (4.2) and (4.3), it follows that

$$\begin{aligned}\Big\|\sum_{j\in\Lambda\cup\{1\}} T_{k_j}(y_j)\Big\|_X &= \Big\|\sum_{j\in\Lambda\cup\{1\}}\sum_{n=1}^{k_j} \tilde{f}_{k_j,2^{k_j}+n}(y_j)\, z_{2^{k_j}+n}\Big\|_X \\ &\leq C_2 \sup_{\substack{1\leq n\leq k_j\\ j\in\Lambda\cup\{1\}}} |\tilde{f}_{k_j,2^{k_j}+n}(y_j)| \leq 2\Big(\frac{C_2}{C_1}\Big)^2\end{aligned}$$

Therefore,

$$\Big\|\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\, x_j\Big\|_X \leq 2\Big(\frac{C_2}{C_1}\Big)^2 + \|T_{k_{\nu(D)}}(x_{\nu(D)})\|_X \leq 3\Big(\frac{C_2}{C_1}\Big)^2$$

wherefrom it follows that $\mathrm{SV}_a^b(F)<\infty$.

In summary, $F\in SV([a,b\,],L(X))$ and F is not simply regulated, which is a contradiction. Thus the lemma is established. $\square$

5 Semivariation and the Kurzweil integral

In the recent years non-absolute integrals have been increasingly investigated. Among them it is worth highlighting the one due to Kurzweil, [26], whose concept of integration has been the background of several papers related to differential and difference equations. See, for instance, [39], [15] and [28].

This section is dedicated to investigate the connection between the semivariation and the integral due to Kurzweil in two different aspects. First, we present a result by Honig which generalizes the following fact: *every function of bounded variation is a multiplier for Kurzweil integrable functions.* Next, we apply the concept of semivariation to derive two convergence results for Stieltjes type integral and we conclude the section by proving a new characterization of semivariation by the means of the abstract Kurzweil-Stieltjes integral.

In what follows we deal with special cases of the integral introduced by J. Kurzweil in [26] under the name "generalized Perron integral". For the reader's convenience, let us recall its definition.

As usual, a *partition of* $[a,b]$ is a tagged division $P=(\tau_j,[\alpha_{j-1},\alpha_j])$ where the set $\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(P)}\}$ is a division of $[a,b]$ and $\tau_j\in[\alpha_{j-1},\alpha_j]$ for $j=1,\dots,\nu(P)$. A *gauge on* $[a,b]$ is a positive function $\delta:[a,b]\to\mathbb{R}^+$. Furthermore, given a gauge δ on $[a,b]$, a partition $P=(\tau_j,[\alpha_{j-1},\alpha_j])$ is called δ-fine if

$$[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))\quad\text{for}\ \ j=1,\dots,\nu(P).$$

Given an arbitrary gauge δ on $[a,b]$, the existence of (at least one) δ-fine partition is a known result, the so-called Cousin's lemma (see [18, Theorem 4.1] or [31, Lemma 1.4]).

A function $U:[a,b]\times[a,b]\to X$ is Kurzweil integrable on $[a,b]$, if there exists $I\in X$ such that for every $\varepsilon>0$, there is a gauge δ on $[a,b]$ such that

$$\left\|\sum_{j=1}^{\nu(P)}[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-I\right\|_X<\varepsilon\ \ \text{for all } \delta\text{-fine partitions of } [a,b].$$

In this case, we define the Kurzweil integral as $\int_a^b DU(\tau,t)=I$.

For a more comprehensive study of the properties of the Kurzweil integral we refer to the monograph [31] and references therein.

Taking $U(\tau,t)=f(\tau)\,t$ for $t\in[a,b]$, where $f:[a,b]\to X$ is a given function, the definition above corresponds to an integration process based on Riemann-type sums, namely, the Henstock-Kurzweil integral. Such integral is known to extend the theory of Lebesgue integral. In what follows, when dealing with the Kurzweil-Henstock integral we will write simply $\int_a^b f(t)\mathrm{d}t$ instead of $\int_a^b D[f(\tau)t]$.

Secondly, we are interested in the abstract Kurzweil-Stieltjes integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b \mathrm{d}[F]\,g$, where $F:[a,b]\to L(X)$ and $g:[a,b]\to X$ (see [33]). These integrals are

obtained from the choices $U(\tau,t) = F(\tau)\,g(t)$ and $U(\tau,t) = F(t)\,g(\tau)$ for $t,\tau \in [a,b\,]$, respectively.

In the sequel we state two existence results for the abstract Kurzweil-Stieltjes integral relatively to functions of bounded semivariation (for the proof see [29, Thereom 3.3]).

Theorem 5.1. *Let $F \in SV([a,b\,],L(X,Y))$.*

(i) If $g \in G([a,b\,],X)$, then the integral $\int_a^b F\,\mathrm{d}[g]$ exists and

$$\Big\|\int_a^b F\,\mathrm{d}[g]\Big\|_X \le \big(\|F(a)\|_X + \|F(b)\|_X + \mathrm{SV}_a^b(F)\big)\,\|g\|_\infty.$$

(ii) If $F \in SG([a,b\,],L(X))$ and $g \in G([a,b\,],X)$, then the integral $\int_a^b \mathrm{d}[F]g$ exists and

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X \le \mathrm{SV}_a^b(F)\,\|g\|_\infty.$$

Let us denote by $\mathcal{K}([a,b\,],X)$ the set of all Henstock-Kurzweil integrable functions, that is, all functions $f:[a,b\,]\to X$ whose integral $\int_a^b f(t)\mathrm{d}t$ exists. The linearity of the integral implies that $\mathcal{K}([a,b\,],X)$ is a linear space (cf. [16] or [31]).

The following theorem, borrowed from [21, 1.15], shows that the functions of bounded semivariation are multipliers for the space $\mathcal{K}([a,b\,],X)$.

Theorem 5.2. *Let $g \in \mathcal{K}([a,b\,],X)$ and $F \in SV([a,b\,],L(X))$. Consider the function $Fg:[a,b\,]\to X$ given by $(Fg)(t) = F(t)g(t)$ for $t \in [a,b\,]$. Then $Fg \in \mathcal{K}([a,b\,],X)$ and*

$$\int_a^b F(t)g(t)\,\mathrm{d}t = \int_a^b F\,\mathrm{d}[\tilde{g}], \tag{5.1}$$

where $\tilde{g}(t) = \int_a^t g(s)\,\mathrm{d}s$ for $t \in [a,b\,]$.

Proof. First of all, noting that the indefinite integral of g defines a continuous function on $[a,b\,]$ (cf. [16, Theorem 9.12]), it is clear by Theorem 5.1 that $\int_a^b F\,\mathrm{d}\tilde{g}$ exists.

Given $\varepsilon > 0$, let δ_1 and δ_2 be gauges on $[a,b\,]$ such that

$$\Big\|\sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde{g}(\alpha_j) - \tilde{g}(\alpha_{j-1})] - \int_a^b F\,\mathrm{d}[\tilde{g}]\Big\|_X < \varepsilon \quad \text{for all } \delta_1\text{-fine partitions of } [a,b\,], \tag{5.2}$$

and

$$\Big\|\sum_{j=1}^{\nu(P)} g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b g(s)\,\mathrm{d}s\Big\|_X < \varepsilon \quad \text{for all } \delta_2\text{-fine partitions of } [a,b\,].$$

Due to the Saks-Henstock Lemma (see [31, Lemma 1.13]), for any δ_2-fine partition of $[a,b]$, $P=(\tau_j,[\alpha_{j-1},\alpha_j])$, we have

$$\Big\| \sum_{k=j}^{\nu(P)} \Big[g(\tau_k)(\alpha_k-\alpha_{k-1}) - \int_{\alpha_{k-1}}^{\alpha_k} g(s)\,\mathrm{d}s \Big] \Big\|_X < \varepsilon \quad \text{for} \;\; j=1,2,\dots,\nu(P). \tag{5.3}$$

Put $\delta(t)=\min\{\delta_1(t),\,\delta_2(t)\}$ for $t\in[a,b]$. Given a δ-fine partition $P=(\tau_j,[\alpha_{j-1},\alpha_j])$ of $[a,b]$, by (5.2) we get

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)g(\tau_j)(\alpha_j-\alpha_{j-1}) - \int_a^b F\,\mathrm{d}[\tilde g] \Big\|_X \\
&\quad \le \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)g(\tau_j)(\alpha_j-\alpha_{j-1}) - \sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde g(\alpha_j)-\tilde g(\alpha_{j-1})] \Big\|_X \\
&\qquad + \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde g(\alpha_j)-\tilde g(\alpha_{j-1})] - \int_a^b F\,\mathrm{d}[\tilde g] \Big\|_X \\
&\quad < \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)\Big[g(\tau_j)(\alpha_j-\alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s \Big] \Big\|_X + \varepsilon
\end{aligned}$$

In order to estimate the other term in the last inequality, we will make use of the following equality mentioned in [21]:

$$\sum_{j=1}^m A_j\,x_j = \sum_{j=1}^m [A_j - A_{j-1}] \left(\sum_{k=j}^m x_k \right) + A_0 \left(\sum_{k=1}^m x_k \right)$$

for all $A_j\in L(X)$ and all $x_j\in X$. Let us consider $m=\nu(P)$ and also

$$A_0 = F(a), \quad A_j = F(\tau_j), \quad x_j = g(\tau_j)(\alpha_j-\alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s$$

for $j=1,\dots\nu(P)$. Note that, by (5.3) we have $\Big\| \sum_{k=j}^{\nu(P)} x_k \Big\|_X \le \varepsilon$ for each $j=1,\dots\nu(P)$. Therefore

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)\Big[g(\tau_j)(\alpha_j-\alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s \Big] \Big\|_X \\
&\quad < \varepsilon \Big\| \sum_{j=1}^{\nu(P)} [F(\tau_j)-F(\tau_{j-1})] \, \frac{\sum_{k=j}^{\nu(P)} x_k}{\varepsilon} \Big\|_X + \Big\| F(a)\Big(\sum_{j=1}^{\nu(P)} x_j \Big) \Big\|_X \\
&\quad < \varepsilon \left(\mathrm{SV}_a^b(F) + \|F(a)\|_{L(X)} \right)
\end{aligned}$$

(where $\tau_0 = a$). Having all these in mind, we obtain

$$\left\| \sum_{j=1}^{\nu(P)} F(\tau_j)g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b F\,\mathrm{d}[\tilde{g}] \right\|_X < \varepsilon\left(1 + \mathrm{SV}_a^b(F) + \|F(a)\|_{L(X)}\right)$$

for all δ-fine partitions of $[a,b]$, wherefrom we conclude that $\int_a^b F(t)\,g(t)$ exists and the unicity of the integral leads to (5.1). $\square$

Remark 5.3. The theorem above is presented in [21] when integration by parts formulas for Henstock-Kurzweil integral are discussed. Indeed, taking into account the results from [36] (see also [29, Corollary 3.6]), the equality (5.1) can be rewritten as

$$\int_a^b F(t)g(t)\,\mathrm{d}t = F(b)\tilde{g}(b) - \int_a^b F\,\mathrm{d}\tilde{g}$$

Moreover, due to the continuity of the function $\tilde{g}$, the Stieltjes-type integral in the formula above (as well as in (5.1)) can be read as a Riemann-Stieltjes integral defined in the Banach space setting (see [21, 1.13]).

We would like also to remark that the result in Theorem 5.2 remains valid if we replace the function $g : [a,b] \to X$ by Henstock-Kurzweil integrable functions defined in $[a,b]$ and taking values in $L(X)$.

Now we turn our attention to the connection between semivariation and the abstract Kurzweil-Stieltjes integral. First, we focus in Helly type result, that is, convergence results for the integral based on assumptions similar to those presented in Lemma 2.4. The theorem in the sequel, to our knowlegde, is not available in literature in the presented formulation.

Theorem 5.4. *Let $F : [a,b] \to L(X)$, a sequence $\{F_n\}_n \subset SV([a,b], L(X))$ and a constant $M > 0$ be such that*

$$\mathrm{SV}_a^b(F_n) \le M \quad \textit{for every} \;\; n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \|F_n(t) - F(t)\|_{L(X)} = 0 \quad \textit{for every} \;\; t \in [a,b].$$

If $g \in G([a,b], X)$, then the integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b F_n\,\mathrm{d}[g]$, $n \in \mathbb{N}$, exist and

$$\lim_{n\to\infty} \int_a^b F_n\,\mathrm{d}[g] = \int_a^b F\,\mathrm{d}[g]. \tag{5.4}$$

Proof. By Lemma 2.4 we know that $F \in SV([a,b], L(X))$, thus the existence of the integrals is guaranteed by Theorem 5.1 (i).

To prove the convergence, we first consider the case when g is a finite step function. Due to the linearity of the integral, it is enough to show that (5.4) holds for functions of the form $\chi_{[a,\tau]}x$, $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$, where $\tau \in (a,b)$ and $x \in X$.

Given $\tau \in [a,b)$ and $x \in X$, by [42, Proposition 2.3.3] (with an obvious extension to Banach spaces-valued functions) we have

$$\int_a^b (F_n - F)\,\mathrm{d}[\chi_{[a,\tau]}x] = F(\tau)x - F_n(\tau)x,$$

hence (5.4) follows. Similarly, one can prove the equality for $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$.

Now, assuming $g \in G([a,b],X)$ and given $\varepsilon > 0$, there exists a finite step function $\varphi : [a,b] \to X$ such that $\|g-\varphi\|_\infty < \varepsilon$ (see [22, Theorem I.3.1]). Let $n_0 \in \mathbb{N}$ be such that

$$\|(F_n - F)(a)\| + \|(F_n - F)(b)\| < M \quad \text{and} \quad \Big\| \int_a^b (F_n - F)\,d[\varphi] \Big\| < \varepsilon$$

for $n > n_0$. These inequalities, together with (2.1) and Theorem 5.1, imply that

$$\begin{aligned}
&\Big\| \int_a^b (F_n - F)\,\mathrm{d}[g] \Big\|_X \\
&\quad\le \Big\| \int_a^b (F_n - F)\,\mathrm{d}[g-\varphi] \Big\|_X + \Big\| \int_a^b (F_n - F)\,\mathrm{d}[\varphi] \Big\|_X \\
&\quad\le \big(\|(F_n - F)(a)\|_X + \|(F_n - F)(b)\|_X + \mathrm{SV}_a^b(F_n - F) \big)\, \|g-\varphi\|_\infty + \varepsilon \\
&\quad< \big(M + \mathrm{SV}_a^b(F_n) + \mathrm{SV}_a^b(F) \big)\,\varepsilon + \varepsilon < \varepsilon(3\,M + 1)
\end{aligned}$$

for every $n > n_0$, which proves (5.4). $\square$

We remark that in [32] the convergence result above is proved for real-valued functions of bounded variation.

Still a Helly type result, the following theorem concerns integrals of the form $\int_a^b \mathrm{d}[F]\,g$.

Theorem 5.5. *Let $F : [a,b] \to L(X)$, $\{F_n\}_n \subset SV([a,b], L(X)) \cap SG([a,b], L(X))$ and a constant $M > 0$ be such that*

$$\mathrm{SV}_a^b(F_n) \le M \quad \textit{for every} \;\; n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \Big(\sup_{t \in [a,b]} \|F_n(t)x - F(t)x\|_X \Big) = 0 \quad \textit{for every} \;\; x \in X. \tag{5.5}$$

If $g \in G([a,b],X)$, then the integrals $\int_a^b \mathrm{d}[F]\,g$ and $\int_a^b \mathrm{d}[F_n]\,g$, $n \in \mathbb{N}$, exist and

$$\lim_{n\to\infty} \int_a^b \mathrm{d}[F_n]\,g = \int_a^b \mathrm{d}[F]\,g. \tag{5.6}$$

Proof. Given $x \in X$, for each $n \in \mathbb{N}$ put $(F_n)_x : t \in [a,b\,] \longmapsto F_n(t)x \in X$. By (5.5) it follows that the sequence of regulated functions $\{(F_n)_x\}_n$ converges uniformly in $[a,b\,]$ to $F_x : t \in [a,b\,] \longmapsto F(t)x \in X$. Hence, by [22, I.3.5] the function F_x is regulated and, since it holds for each $x \in X$, we have $F \in SG([a,b\,], L(X))$. Noting that $F \in SV([a,b\,], L(X))$ (see Lemma 2.4), the existence of the integral $\int_a^b \mathrm{d}[F]\, g$, as well as the existence of $\int_a^b \mathrm{d}[F_n]\, g$, $n \in \mathbb{N}$, yields from Theorem 5.1 (*ii*).

In order to prove (5.6), we first consider the case when $g(t) = \chi_{[a,\tau]}(t)\tilde{x}$ for $t \in [a,b\,]$, where $\tau \in (a,b)$ and $\tilde{x} \in X$ are arbitrarily fixed. For each $n \in \mathbb{N}$ we have

$$\int_a^b \mathrm{d}[F_n - F]g = \lim_{s\to\tau-} F_n(s)\tilde{x} - \lim_{s\to\tau-} F(s)\tilde{x} - [F_n(a) - F(a)]\tilde{x},$$

(cf. [33, Proposition 14]) or equivalently,

$$\int_a^b \mathrm{d}[F_n - F]g = F_n(\tau\dot{-})\tilde{x} - F(\tau\dot{-})\tilde{x} - [F_n(a) - F(a)]\tilde{x}, \tag{5.7}$$

where $F_n(\tau\dot{-})$, $F(\tau\dot{-}) \in L(X)$ are operators satisfying

$$\lim_{s\to\tau-} F_n(s)x = F_n(\tau\dot{-})x \quad \text{and} \quad \lim_{s\to\tau-} F(s)x = F(\tau\dot{-})x$$

for every $x \in X$. Given $\varepsilon > 0$, by (5.5) there exists $n_0 \in \mathbb{N}$ such that

$$\|[F_n(t) - F(t)]\tilde{x}\|_X < \frac{\varepsilon}{3} \quad \text{for} \ \ n \geq n_0 \ \ \text{and} \ \ t \in [a,b\,]. \tag{5.8}$$

Fixed $n \geq n_0$, we can choose $\delta > 0$ such that

$$\|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X \leq \frac{\varepsilon}{3} \quad \text{and} \quad \|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X \leq \frac{\varepsilon}{3}. \tag{5.9}$$

Let us choose $s \in (\tau - \delta, \tau)$. From (5.8) and (5.9) it follows that

$$\begin{aligned}
&\|F_n(\tau\dot{-})\tilde{x} - F(\tau\dot{-})\tilde{x}\|_X \\
&\qquad \leq \|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X + \|F_n(s)\tilde{x} - F(s)\tilde{x}\|_X + \|F(s)\tilde{x} - F(\tau\dot{-})\tilde{x}\|_X < \varepsilon,
\end{aligned}$$

which together with (5.8) applied to $t = a$, shows that the integral in (5.7) tends to zero. With similar argument we can prove that (5.6) holds when g is a function of the form $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$ for $\tau \in (a,b)$ and $x \in X$. As a consequence of the linearity of the integral we conclude that (5.6) is valid if g is a step function.

Now, assuming that $g \in G([a,b\,], X)$ and given $\varepsilon > 0$, let $\varphi : [a,b\,] \to X$ be a finite step function such that $\|g - \varphi\|_\infty < \varepsilon$ (see [22, Theorem I.3.1]). Thus, by Theorem 5.1

(i) we have

$$\Big\| \int_a^b \mathrm{d}[F_n - F]\, g \Big\|_X \le \Big\| \int_a^b \mathrm{d}[F_n - F]\, (g - \varphi) \Big\|_X + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X$$
$$\le \mathrm{SV}_a^b(F_n - F) \|g - \varphi\|_\infty + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X$$
$$\le 2\, M\varepsilon + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X .$$

Since φ is a step function, the result now follows from first part of the proof. $\square$

Similar convergence results have been proved in [30] and [28] in the frame of functions of bounded variation.

In literature the notion of variation is sometimes described by the means of different integrals of the Stieltjes type. In [5], using the Young integral on Hilbert spaces, not only a characterization for the norm $\|\cdot\|_{BV}$ is presented but also the notion of essential variation is treated. Dealing with the semivariation and the interior integral (i.e. the Dushnik integral), it is worth highlighting [22, Corollary I.5.2].

Inspired by those results, we present here a characterization of the semivariation via the abstract Kurzweil-Stieltjes integral. To this end, we will need the following estimates whose proofs are quite similar to [29, Lemma 3.1].

Lemma 5.6. *Let $F : [a, b\,] \to L(X)$ and $g : [a, b\,] \to X$ be given. For every partition $P = (\tau_j, [\alpha_{j-1}, \alpha_j])$ of $[a, b\,]$ we have*

$$\Bigg\| F(b)\, g(b) - \sum_{j=1}^{\nu(P)} F(\tau_j)[g(\alpha_j) - g(\alpha_{j-1})] \Bigg\|_X \le \|F(a)\, g(a)\|_X + \|g\|_\infty\, \mathrm{SV}_a^b(F),$$

Furthermore, if $\int_a^b F\,\mathrm{d}[g]$ exists then

$$\Big\| F(b)\, g(b) - \int_a^b F\,\mathrm{d}[g] \Big\|_X \le \|F(a)\, g(a)\|_X + \|g\|_\infty\, \mathrm{SV}_a^b(F). \tag{5.10}$$

Now we present the main result of this section. In what follows, $S_L([a, b\,], X)$ denotes the set of all finite step functions $g : [a, b\,] \to X$ which are left-continuous on $(a, b]$ and such that $g(a) = 0$.

Theorem 5.7. *If $F \in SV([a, b\,], L(X))$, then*

$$\mathrm{SV}_a^b(F) = \sup \left\{ \Big\| F(b)\, g(b) - \int_a^b F\,\mathrm{d}[g] \Big\|_X ;\ g \in S_L([a, b\,], X),\ \|g\|_\infty \le 1 \right\}.$$

Proof. At first, note that by (5.10) $\mathrm{SV}_a^b(F)$ is an upper bound to the set

$$\mathcal{A} := \left\{ \Big\| F(b)\, g(b) - \int_a^b F \,\mathrm{d}[g] \Big\|_X ; \, g \in S_L([a,b],X),\, g(a) = 0 \;\text{ and }\; \|g\|_\infty \leq 1 \right\}.$$

To conclude the proof it is enough to show that $\mathrm{SV}_a^b(F) \leq \sup \mathcal{A}$.

Let $\varepsilon > 0$ be given. Then, there exist a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ of $[a,b]$ and $x_j \in X$, $j = 1, \ldots, m$ with $\|x_j\| \leq 1$ such that

$$\mathrm{SV}_a^b(F) - \varepsilon < \Big\| \sum_{j=1}^m [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_X.$$

Let $\tilde{g} : [a,b] \to X$ be the function given by

$$\tilde{g}(t) = \sum_{j=1}^m \chi_{(\alpha_{j-1},\alpha_j]}(t)\, x_j \quad \text{for} \;\; t \in [a,b].$$

Thus, $\tilde{g}$ is a left continuous step function with $\tilde{g}(a) = 0$ and $\|\tilde{g}\|_\infty \leq 1$, that is, $\tilde{g} \in S_L([a,b],X)$. Calculating the integral $\int_a^b F \,\mathrm{d}[\tilde{g}]$, we have

$$\int_a^b F \,\mathrm{d}[\tilde{g}] = -\sum_{j=1}^{m-1} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j + F(\alpha_{m-1})\, x_m$$

(see [33, Proposition 14]). Therefore,

$$\mathrm{SV}_a^b(F) - \varepsilon < \Big\| \sum_{j=1}^m [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_X = \Big\| F(b)\, \tilde{g}(b) - \int_a^b F \,\mathrm{d}[\tilde{g}] \Big\|_X \leq \sup \mathcal{A}.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

A Appendix: Series in Banach space

Definition A.1. Let $x_n \in X$ for $n \in \mathbb{N}$. We say that:

1. The series $\displaystyle\sum_{n=1}^\infty x_n$ is convergent if the sequence of its partial sums $s_n = \displaystyle\sum_{k=1}^n x_k$ converges in X.
2. The series $\displaystyle\sum_{n=1}^\infty x_n$ is absolutely convergent if $\displaystyle\sum_{n=1}^\infty \|x_n\|_X < \infty$.

3. The series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent if the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges in X for any permutation π of $\mathbb{N}$.

Theorem A.2. *If* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent, then all rearrangements have the same sum.*

(See [24, Theorem 1.3.1])

Theorem A.3. *For series* $\sum_{n=1}^{\infty} x_n$ *in* X *the following conditions are equivalent:*

(a) the series is unconditionally convergent;

(b) for any bounded sequence $\{\alpha_n\}_n$ *in* $\mathbb{R}$*, the series* $\sum_{n=1}^{\infty} \alpha_n x_n$ *converges in* X*.*

(See [24, Theorem 1.3.2] or [1, Proposition 2.4.9])

Lemma A.4. *If the series* $\sum_{n=1}^{\infty} x_n$ *is unconditittionally convergent in* X*, then*

$$\left\{ \sum_{n=1}^{\infty} \alpha_n \, x_n \; : \; \alpha_n \in \mathbb{R} \quad \textit{with} \quad |\alpha_n| \leq 1, \, n \in \mathbb{N} \right\}$$

is a bounded subset of X*.*

(See [40, Lemma 1])

It is clear that absolute convergence implies unconditional convergence. However, the converse is not true in general.

Theorem A.5. (Dvoretzky-Rogers) *If every unconditionally convergent series in an Banach space* X *is absolutely convergent, then the dimension of* X *is finite.*

(See [12], [1, Theorem 8.2.14] or [8, Chapter VI])

Corollary A.6. *In every infinite-dimensional Banach space there exists an unconditionally convergent series that is not absolutely convergent.*

In the sequel we recall some aspects of convergence of series involving weak topology.

Definition A.7. Let $x_n \in X$ for $n \in \mathbb{N}$. We say that:

1. The sequence $\{x_n\}_n$ is a weakly Cauchy sequence if the sequence $\{x^*(x_n)\}_n$ converges in $\mathbb{R}$ for every $x^* \in X^*$
2. The series $\sum_{n=1}^{\infty} x_n$ is weakly convergent if there exists $z \in X$ such that the series $\sum_{n=1}^{\infty} x^*(x_n)$ converges to $x^*(z)$ for every $x^* \in X^*$.
3. The series $\sum_{n=1}^{\infty} x_n$ is weakly absolutely convergent if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$.

Proposition A.8. *If the series* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent, then it is weakly absolutely convergent.*

(See [1, Proposition 2.4.4 (*iii*)])

The converse of Proposition A.8 charaterizes in an important class of Banach spaces.

Theorem A.9. (Bessaga-Pelczynski) *A Banach space X does not contain an isomorphic copy of c_0 if and only if every weakly absolutely convergent series in X is unconditionally convergent.*

(See [8, Theorem V.8], [24, Theorem 6.4.3] or [1, Theorem 2.4.11])

Another important class of spaces which is worth mentioning is the class of weakly sequentially complete Banach spaces. Recall that X is weakly sequentially complete if every weakly Cauchy sequence is weakly convergent in X. In which concerns series in such spaces, we have the following result.

Theorem A.10. *If X is weakly sequentially complete, then every weakly unconditionally convergent series is unconditionally convergent in X.*

(See [19, Theorem 3.2.3] or [1, Corollary 2.4.15])

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